

The Replica Limit of Unitary Matrix Integrals

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Abstract

We investigate the replica trick for the microscopic spectral density, $\rho_s(x)$, of the Euclidean QCD Dirac operator. Our starting point is the low-energy limit of the QCD partition function for n fermionic flavors (or replicas) in the sector of topological charge ν . In the domain of the smallest eigenvalues, this partition function is simply given by a $U(n)$ unitary matrix integral. We show that the asymptotic behavior of $\rho_s(x)$ for $x \rightarrow \infty$ is obtained from the $n \rightarrow 0$ limit of this integral. The smooth contributions to this series are obtained from an expansion about the replica symmetric saddle-point, whereas the oscillatory terms follow from an expansion about a saddle-point that breaks the replica symmetry. For $\nu = 0$ we recover the small- x logarithmic singularity of the resolvent by means of the replica trick. For half integer ν , when the saddle point expansion of the $U(n)$ integral terminates, the replica trick reproduces the exact analytical result. In all other cases only an asymptotic series that does not uniquely determine the microscopic spectral density is obtained. We argue that bosonic replicas fail to reproduce the microscopic spectral density. In all cases, the exact answer is obtained naturally by means of the supersymmetric method.

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1 Introduction

In disordered systems, the ensemble average of the logarithm of the partition function cannot be evaluated directly in most cases. Two widely used methods, the replica trick [1] and the supersymmetric method [2, 3, 4], have been proposed to circumvent this problem. In the replica trick the ensemble average of the logarithm of the partition function is written as

$$\langle \log Z \rangle = \lim_{n \rightarrow 0} \left\langle \frac{Z^n - 1}{n} \right\rangle, \quad (1)$$

i.e., $\log Z$ is calculated by the analytic continuation of the n -dependence of n replicated partition functions to $n \rightarrow 0$. If Z is given by a fermion determinant, Z^n can be written as an integral over n replicated Grassmann fields, and therefore, this method is known as the fermionic replica trick. Alternatively, $\log Z$ can be written as

$$\langle \log Z \rangle = \lim_{n \rightarrow 0} \left\langle \frac{1 - Z^{-n}}{n} \right\rangle. \quad (2)$$

In this case, a determinantal Z can be expressed as an integral over n replicated complex fields and this limit referred to as the bosonic replica trick. Both for the fermionic and for the bosonic replica trick, the average partition function can be mapped onto a non-linear σ -model which is amenable to a saddle-point expansion.

In the supersymmetric method the disorder average is performed for the ratio

$$\langle \partial_J \log Z(J) \rangle_{J=0} = \partial_J \left\langle \frac{Z(J)}{Z(J=0)} \right\rangle \Big|_{J=0}. \quad (3)$$

If $Z(J)$ is given by a determinant, the numerator can be expressed as a fermionic integral whereas the denominator can be written as a bosonic integral. For $J = 0$ the partition function is thus invariant with respect to superunitary transformations that mix the fermionic and bosonic fields. Based on this symmetry, the supersymmetric partition function can be mapped onto a supersymmetric non-linear σ -model [3, 4] which can be used to derive exact analytical expressions for spectral correlation functions.

The replica trick has two obvious advantages. First, it can be applied to cases where Z cannot be expressed as a determinant (as for example is the case in the theory of spin-glasses [5]), and, second, it is possible to calculate the logarithm of the partition function. One disadvantage of this method is that, in order to take the replica limit, the n dependence of the replicated partition function has to be known analytically. Therefore, the application of replica trick is limited to perturbative expansions of the partition function. A much more serious problem of this method is that the continuation of the n -dependence to $n \rightarrow 0$ is not unique. For example, a term of the form $\sim \sin n\pi$ contributing to Z^n vanishes for integer n but gives rise to a nonzero result in the replica limit. The failure of the

replica trick was first noticed in the theory of spin glasses [5] where a replica symmetric minimum of the free energy resulted in a negative entropy [6]. However, in that case the problems could be resolved by means of an elaborate scheme of replica symmetry breaking [6]. More recently, the replica trick was criticized because of its failure to reproduce the oscillatory contributions to random matrix correlation functions [7]. Notice however that the non-oscillatory contributions to the two-point function were reproduced correctly [8, 9]. Another example for which the replica trick may be problematic are nonhermitian Random Matrix Theories with eigenvalues scattered in the complex plane [10], but we will not study such theories in this article.

The advantage of the supersymmetric method is that it is possible to derive non-perturbative analytical results. This has been shown convincingly for the calculation of spectral correlation functions of random matrix-ensembles [3, 4, 7, 11, 12]. A disadvantage is that it requires some familiarity with supermathematics, but for perturbative expansions this method is no more complicated than the replica trick. In the example that will be discussed in this article, the exact super-symmetric calculation is actually much simpler than the perturbative replica calculation. A second disadvantage is, that because the average partition function is normalized to unity, one does not have access to the average free energy.

Recently, the replica trick was revived in an article by Kamenev and Mézard [13]. They found that, in order to reproduce the oscillatory terms in spectral two-point correlation function, saddle points with broken replica symmetry had to be taken into account [13]. For the Gaussian Unitary Ensemble they found the exact analytical result. As explained in an article by Zirnbauer [14], the reason for this miracle is a consequence of the Duistermaat-Heckman theorem [15] which is applicable to the σ -model for the two-point function of the Gaussian Unitary Ensemble. This theorem states the conditions under which an integral is localized on its critical points so that a saddle-point approximation becomes exact. For the Gaussian Orthogonal Ensemble and the Gaussian Symplectic Ensemble the replica trick could only reproduce the asymptotic expansion of the two-point spectral correlation function for large energy differences [16, 17]. We remind the reader that, unless we know the analytical properties of a function in the complex plane, it cannot be reproduced from its asymptotic series. A version of the replica trick that does not rely on the non-linear σ -model but instead on orthogonal polynomials was shown to reproduce the exact correlation functions for all Gaussian ensembles [18]. In this article we will not discuss this variant of the replica trick which is not an alternative to the orthogonal polynomial method.

To investigate the replica trick we have chosen the microscopic spectral density of the QCD Dirac operator which is defined as the spectral density near zero on the scale of the average level spacing. The reason is three-fold. i) The n -fold replicated partition

function is the QCD partition function with n flavors. Its low-energy limit, relevant for the microscopic spectral density, is known analytically. Because of spontaneous breaking of chiral symmetry in QCD, it is given by a partition function of weakly interacting Goldstone bosons and, on the scale of the average level spacing, it can be reduced to a unitary matrix integral which can be evaluated analytically for any number of flavors [19, 20, 22]. This, so called finite volume chiral partition function has been investigated in great detail, also in the context of one-link integrals in lattice QCD [19, 23]. ii) Because the eigenvalues occur in pairs $\pm\lambda$, the level repulsion of the eigenvalues leads to a nontrivial oscillatory behavior of the microscopic spectral density. iii) The low energy partition function can be derived as a function of two integer valued parameters, the topological charge ν and the number of physical flavors N_f (in this article only $N_f = 0$ will be discussed) and can be trivially continued to non-integer ν . For half-integer ν , the saddle point expansion of the $U(n)$ -integrals terminates for finite positive integer values of n . This is closely related to Duistermaat-Heckman localization, where the leading order saddle point approximation is exact such as for the σ -model with n fermionic replicas of the two-point function of the Gaussian Unitary Ensemble [14]. Based on recent work [13, 14], we expect that the replica trick with replica-symmetry breaking gives the exact result in this case.

The replica limit of the finite volume chiral partition function was first studied in [24]. It was found that the asymptotic expansion of the valence quark mass dependence of the chiral condensate in the microscopic region (i.e. for large valence quark masses in units of the average eigenvalue spacing) was reproduced by the replica trick. The small mass expansion was obtained up to logarithmic singularities which are essential for the calculation of the spectral density. Recently, the partially quenched supersymmetric chiral Lagrangian [26, 11, 12, 27] was formulated in terms of the replica trick [25].

In this article we analyze the oscillatory contributions to the microscopic spectral density in the framework of the replica trick. In section 2 we discuss the chiral symmetries for bosonic replicas and introduce the supersymmetric quenched low energy chiral partition function. The calculation of the microscopic spectral density by means of the supersymmetric method is given in section 3. This calculation illustrates that the compact/non-compact structure of the final result for the resolvent appears naturally in the supersymmetric method. In section 4.1 we derive the large-mass asymptotic behavior of the microscopic spectral density by means of the fermionic replica trick. The small-mass behavior is discussed in section 4.2. Bosonic replicas are discussed in section 5 and concluding remarks are made in section 6. In the Appendix we derive trace correlators necessary for the replica calculation to fourth order in the inverse mass.

2 Chiral Symmetry

In this section we discuss chiral symmetry for bosonic and fermionic replicas, as well as for the supersymmetric partition function. To illustrate the difference in flavor symmetries between bosonic and fermionic replicas we present a detailed discussion for the case of one flavor or replica.

The question we wish to address in this article is whether the spectrum of the QCD Dirac operator in the sector of topological charge ν can be obtained from the QCD partition function with n additional replica flavors with quark mass z . This partition function is given by

$$Z_\nu^{(N_f+n)}(z) = \int [dA]_\nu \det^n(i\mathcal{D} + z) \prod_{f=1}^{N_f} \det(i\mathcal{D} + m_f) e^{-S_{YM}[A]}, \quad (4)$$

where m_1, \dots, m_{N_f} are the usual quark masses. The integral is over all gauge fields in the sector of topological charge ν (which is chosen positive in this article) and is weighted by the Yang-Mills action. The replica limit of the resolvent or the chiral condensate is defined by

$$\Sigma(z) \equiv \frac{1}{V_4} \text{Tr} \left\langle \frac{1}{z + i\mathcal{D}} \right\rangle = \lim_{n \rightarrow 0} \frac{1}{n} \frac{1}{V_4} \frac{\partial}{\partial z} \ln Z_\nu^{(N_f+n)}(z). \quad (5)$$

Here, V_4 is the Euclidean 4-volume. The spectral density, which in terms of the eigenvalues $i\lambda_k$ of the Dirac operator is defined by

$$\rho(\lambda) = \sum_k \delta(\lambda - \lambda_k), \quad (6)$$

follows from the discontinuity across the imaginary axis

$$\frac{\rho(\lambda)}{V_4} = \frac{1}{2\pi} [\Sigma(i\lambda + \epsilon) - \Sigma(i\lambda - \epsilon)]. \quad (7)$$

Below we only consider the dimensionless ratio

$$\begin{aligned} \frac{\Sigma(z)}{\Sigma_0} &= \frac{1}{V_4 \Sigma_0} \int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda)}{z + i\lambda}, \\ &= \int_{-\infty}^{\infty} du \frac{1}{V_4 \Sigma_0} \rho\left(\frac{u}{V_4 \Sigma_0}\right) \frac{1}{V_4 \Sigma_0 z + iu}, \end{aligned} \quad (8)$$

as a function of the microscopic variable $V_4 \Sigma_0 z$. The chiral condensate, Σ_0 , is defined as the limit of $\Sigma(z)$ for z close to $z = 0$ but many level spacings away from the center of the spectrum. As one can see from the second equality, it can be expressed as an integral over the microscopic spectral density defined by [28]

$$\rho_s(u) = \frac{1}{V_4 \Sigma_0} \rho\left(\frac{u}{V_4 \Sigma_0}\right). \quad (9)$$

For spontaneously broken chiral symmetry, the spacing of the eigenvalues is given by $\pi/\Sigma_0 V_4$ so that (9) is stable in V_4 and can be calculated in the thermodynamic limit.

Because the resolvent $\Sigma(z)$ has a cut along the imaginary axis it is sometimes more convenient to use the relation $G(i\lambda - \epsilon) = -G(-i\lambda + \epsilon)$ to rewrite the discontinuity as

$$\frac{\rho(\lambda)}{V_4} = \frac{1}{2\pi} [\Sigma(i\lambda + \epsilon) + \Sigma(-i\lambda + \epsilon)]. \quad (10)$$

We thus only need to calculate the resolvent in the half-plane $\text{Re}(z) > 0$. If the argument $\Sigma(z)$ represents the microscopic variable, this relation gives us the microscopic spectral density.

The reason of working with the partition function (4) is that in the phase of spontaneously broken symmetry its low energy limit is entirely determined by chiral symmetry and is a partition function of weakly interacting Goldstone modes (or pions). We are interested in the kinematical domain [20]

$$1/m_\pi \gg V_4^{1/4} \gg 1/\Lambda_{QCD}. \quad (11)$$

Because $V_4^{1/4} \gg 1/\Lambda_{QCD}$, only the Goldstone modes contribute to the mass-dependence of the partition function [21]. For quark masses for which the Compton wavelength of the Goldstone modes is much larger than the size of the box ($1/m_\pi \gg V_4^{1/4}$), the kinetic term of the chiral Lagrangian can be ignored, and only the constant fields contribute to the mass dependence of the low-energy partition function. Therefore, as we will see next, in the domain (11) the QCD partition function can be reduced to a unitary matrix integral.

In QCD with n fermionic flavors the chiral symmetry group is given by $U_V(n) \times U_A(n)$. A $U_A(1)$ subgroup of the axial symmetry group is broken by the anomaly. The remaining axial symmetry group is broken spontaneously by the formation of a nonzero chiral condensate and the vector symmetry group, $U_V(n)$, remains unbroken. The Goldstone manifold is thus given by the axial group $SU_A(n)$. The low energy limit of the QCD partition function is uniquely fixed from the requirement that its transformation properties under the chiral symmetry group are the same as for full QCD. Taking into account the anomaly, one finds, for n flavors all with mass m , in the sector of topological charge ν , the low-energy finite volume partition function [20]

$$Z_\nu^{(n)}(x) = \int_{U \in U(n)} dU (\det U)^\nu e^{\frac{x}{2} \text{Tr}(U + U^{-1})}, \quad (12)$$

where $x \equiv mV_4\Sigma_0$. This partition function is valid in the range (11).

In the domain (11), the low energy partition function can also be obtained from a Random Matrix Theory with the global symmetries of the QCD partition function [28]. In this theory, the matrix elements of the Euclidean Dirac operator are replaced by

independently distributed Gaussian random variables. In the sector of topological charge ν , the Dirac matrix is thus has given by

$$D = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix}, \quad (13)$$

where W is an $n \times (n + \nu)$ matrix, and the integration over the gauge fields is replaced by an integration over the probability distribution of the matrix elements. If W is complex and the probability distribution is a function of traces $\text{Tr}(W^\dagger W)^p$, this ensemble is known as the chiral Unitary Ensemble (chUE) or the chiral Gaussian Unitary Ensemble (chGUE) if the distribution of the matrix elements is Gaussian. As was shown in [29], the statistical properties of the smallest eigenvalues of D do not depend on the details of the probability distribution of the matrix elements.

In order to analyze the chiral symmetry for bosonic replicas let us first discuss the flavor symmetries for one flavor. The fermion determinant that occurs in the QCD partition function can be written as a integral over Grassmann variables.

$$\det(D + m) = \int d\bar{\chi} d\chi e^{-\int d^4x \bar{\chi}(D+m)\chi}. \quad (14)$$

The inverse determinant can be written as an integral over bosonic integration variables

$$\frac{1}{\det(D + m)} = \int d\phi^* d\phi e^{-\int d^4x \phi^*(D+m)\phi}. \quad (15)$$

In a well-defined theory the functional integral has to be convergent. This is automatically the case for the Grassmann integration in (14), but the bosonic integrals in (15) is only convergent for positive m . The symmetries of the partition function should be compatible with these convergence requirements. In particular, ϕ^* should be identified with the complex conjugate of ϕ , and not as an independent integration variable such as the fermionic variables $\bar{\chi}$ and χ .

If we decompose the spinors according to the block structure of the Dirac operator the vector symmetry of the massive theory with one fermionic replica (14) is given by

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \rightarrow e^{i\theta} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \begin{pmatrix} \bar{\chi}_1 \\ \bar{\chi}_2 \end{pmatrix} \rightarrow e^{-i\theta} \begin{pmatrix} \bar{\chi}_1 \\ \bar{\chi}_2 \end{pmatrix}, \quad (16)$$

and the axial $U(1)$ symmetry of the massless theory can be written as

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\theta} \chi_1 \\ e^{-i\theta} \chi_2 \end{pmatrix}, \quad \begin{pmatrix} \bar{\chi}_1 \\ \bar{\chi}_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\theta} \bar{\chi}_1 \\ e^{-i\theta} \bar{\chi}_2 \end{pmatrix}. \quad (17)$$

The $U(1)$ vector symmetry of the massive bosonic theory (15) is the same as for the fermionic theory

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow e^{i\theta} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \rightarrow e^{-i\theta} \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix}. \quad (18)$$

This transformation does not affect the complex conjugation properties of ϕ . However, the axial transformation (17) applied to the bosonic fields affects their complex conjugation properties. In this case the axial transformation that is compatible with the convergence of the bosonic integral is given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^s \phi_1 \\ e^{-s} \phi_2 \end{pmatrix}, \quad \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \rightarrow \begin{pmatrix} e^s \phi_1^* \\ e^{-s} \phi_2^* \end{pmatrix}. \quad (19)$$

The axial symmetry group is therefore not $U(1)$ but instead $Gl(1)/U(1)$. Of course, this axial transformation is also a symmetry of the fermionic partition function.

For n bosonic flavors the vector symmetry is $U(n)$ whereas the axial symmetry is given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^H \phi_1 \\ e^{-H} \phi_2 \end{pmatrix}, \quad \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \rightarrow \begin{pmatrix} e^H \phi_1^* \\ e^{-H} \phi_2^* \end{pmatrix}, \quad (20)$$

with the matrix H containing only real elements. The axial symmetry group is thus given by the coset $Gl(n)/U(n)$. An explicit parameterization of this coset is given by AA^\dagger with $A \in Gl(n)$. We expect that the axial symmetry of the bosonic partition function

$$Z_\nu^{(-n)} = \left\langle \frac{1}{\det^n(D+m)} \right\rangle_\nu \quad (21)$$

is broken in the same way as in the fermionic case with a $Gl(1)/U(1)$ coset broken explicitly by the anomaly and the remaining part of the coset broken spontaneously by the chiral condensate. In absence of explicit symmetry breaking, the Goldstone manifold is thus given by $Gl(n)/U(n)$.

The mass term introduced according to

$$\sum_{k,l=1}^n \phi_1^{*k} M_{kl} \phi_1^l + \phi_2^{*k} M_{kl}^\dagger \phi_2^l \quad (22)$$

is invariant under the axial transformation (20) provided that the mass matrix is transformed at the same time as

$$M \rightarrow e^{-H} M e^{-H}, \quad M^\dagger \rightarrow e^H M^\dagger e^H. \quad (23)$$

The bosonic partition function in the sector of topological charge ν transforms covariantly with (20)

$$Z_\nu^{(-n)} \rightarrow \det(e^{2\nu H}) Z_\nu^{(-n)}. \quad (24)$$

The low energy limit of bosonic partition function is uniquely fixed by the requirement that its transformation properties are the same as of $Z_\nu^{(-n)}$. In the sector of topological charge ν it is given by

$$Z_\nu^{(-n)} = \int_{U \in Gl(n)/U(n)} dU (\det U)^\nu e^{\frac{\Sigma_0 V_4}{2} \text{Tr}(MU + M^\dagger U^{-1})}, \quad (25)$$

The measure dU is the invariant Haar measure. Below we consider only the case with a diagonal mass matrix with all nonzero matrix elements equal to m and use the definition $x = mV_4\Sigma_0$. This partition function is valid in the kinematical domain (11) where constant Goldstone fields are the only relevant degrees of freedom.

Since fermionic integrals are always convergent the partition function is invariant under both compact ($U(n)$) and non-compact ($Gl(n)/U(n)$) axial transformations. However, the small mass behavior of the non-compact partition function is singular because of the volume of the non-compact group diverges. This is not the case for the fermionic partition function and therefore this $Gl(n)/U(n)$ is not an admissible parameterization of the Goldstone manifold. The correct parameterization is given by the compact manifold $U(n)$. At a more technical level, this follows from the fact that the transformations that lead to the non-compact integral are only legitimate for bosonic quarks. For fermionic quarks one necessarily finds a compact effective partition function.

In the supersymmetric method the generating function of the quenched resolvent is given by

$$Z_\nu(z, J) = \left\langle \frac{\det(\not{D} + z + J)}{\det(\not{D} + z)} \right\rangle_\nu. \quad (26)$$

It can be written as a superintegral

$$Z_\nu(M, M^\dagger) = \left\langle \int d\phi d\phi^* d\chi d\bar{\chi} \exp \left[\phi^* \not{D} \phi + \bar{\chi} \not{D} \chi + \begin{pmatrix} \phi_1^* \\ \bar{\chi}_1 \end{pmatrix} M \begin{pmatrix} \phi_1 \\ \chi_1 \end{pmatrix} + \begin{pmatrix} \phi_2^* \\ \bar{\chi}_2 \end{pmatrix} M^\dagger \begin{pmatrix} \phi_2 \\ \chi_2 \end{pmatrix} \right] \right\rangle_\nu. \quad (27)$$

Both M and M^\dagger are given by $\text{diag}(z + J, z)$, but in order to study the transformation properties of the partition function, we keep them as general matrices.

With spontaneously broken axial symmetry, the bosonic part of the Goldstone manifold is $U(1) \times Gl(1)/U(1)$. Because, the partition function is invariant under super-unitary transformations, the full symmetry group is given by the maximum Riemannian submanifold of $Gl(1|1)$ [30, 11, 12] (we will denote this manifold by $(\hat{G}l(1|1))$). This manifold can be parameterized as

$$U = \begin{pmatrix} e^{i\theta} & \alpha \\ \beta & e^s \end{pmatrix}. \quad (28)$$

For zero topological charge the generating function (26) is invariant under $Gl_R(1|1) \times Gl_L(1|1)$ if at the same time the mass matrix is transformed as

$$M \rightarrow U_R^{-1} M U_L, \quad M^\dagger \rightarrow U_L^{-1} M^\dagger U_R. \quad (29)$$

For nonzero values of ν the generating function (26) is not invariant under (29) but transforms according to

$$Z_\nu(z, J) \rightarrow \text{Sdet}^\nu(U_R^{-1} U_L) Z_\nu(z, J). \quad (30)$$

The low-energy partition function is obtained from the requirement that it should have the same transformation properties as the QCD partition function (26). In the sector of topological charge ν , it is given by

$$Z_\nu(z, J) = \int_{U \in \hat{G}l(1|1)} dU \text{Sdet}^\nu(U) e^{\frac{\Sigma_0 V_4}{2} \text{Str}(MU + M^\dagger U^{-1})}. \quad (31)$$

The integration is over the Haar measure of $\hat{G}l(1|1)$. Below, we only consider the case of a diagonal mass matrix with both M and M^\dagger equal to $\text{diag}(z + J, z)$.

An amusing observation is that the topological charge in QCD partition function is discrete, whereas the number of flavors with equal mass, thought of as the power of the fermion determinant is a continuous parameter. In the low energy partition function it is just the other way round.

3 Supersymmetric Calculation of the Resolvent

In this section we calculate the super-integrals in (34) to obtain analytical expression for the resolvent and the microscopic spectral density. For integer ν , this calculation was also presented in [11, 12]. To simplify the explicit calculations we will choose the parameterization

$$U = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^s \end{pmatrix} \exp \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}. \quad (32)$$

In this case the Haar measure is simply given by

$$dU = d\theta ds d\alpha d\beta. \quad (33)$$

This results in the partition function

$$Z(z, z + J) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{C_c} d\theta d\alpha d\beta e^{i\nu\theta - \nu s} \exp \text{Str} M \begin{pmatrix} (1 + \frac{\alpha\beta}{2}) \cos \theta & \alpha(e^s - e^{-i\theta}) \\ \beta(e^{i\theta} - e^{-s}) & (1 - \frac{\alpha\beta}{2}) \cosh s \end{pmatrix}, \quad (34)$$

Where z and J are now microscopic variables, i.e. they are expressed in units of $1/V_4 \Sigma_0$. The integration over s is over the complete real axis. For integer ν the integration over θ is over the interval $[-\pi, \pi]$. For non-integer ν the translational invariance of the θ -integral is lost. It is recovered by extending the integration contour to include the intervals $\langle -\pi + i\infty, -\pi \rangle$ and $\langle \pi, \pi + i\infty \rangle$. A picture of this integration contour, denoted by C_c , is shown in Fig. 1.

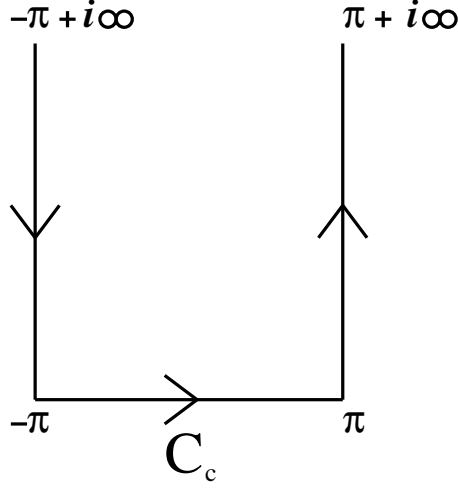


Figure 1: Contour for the θ integration.

After performing the Grassmann integrations the partition function reduces to

$$\begin{aligned}
Z(z, J) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{C_c} d\theta e^{i\nu\theta - \nu s} ((z + J) \cos \theta + z \cosh s) e^{(z+J) \cos \theta - z \cosh s} \\
&= (z + J) K_{\nu}(z) I_{\nu+1}(z + J) + z K_{\nu+1}(z) I_{\nu}(z + J).
\end{aligned} \tag{35}$$

The normalization of the partition function according to $Z(z, J = 0) = 1$ follows from the Wronskian identity $z K_{\nu}(z) I_{\nu+1}(z) + z K_{\nu+1}(z) I_{\nu}(z) = 1$. The resolvent obtained by differentiation with respect to J is, after using some identities for Bessel functions, given by

$$\Sigma(z) = z(K_{\nu}(z) I_{\nu}(z) + K_{\nu-1}(z) I_{\nu+1}(z)) + \frac{\nu}{z}. \tag{36}$$

The microscopic spectral density then follows from the discontinuity across the imaginary axis according to (10),

$$\begin{aligned}
\rho_s(x) &= \frac{1}{2\pi} [\Sigma(ix + \epsilon) + \Sigma(-ix + \epsilon)] \\
&= \frac{x}{2} (J_{\nu}(x)^2 - J_{\nu+1}(x) J_{\nu-1}(x)) + \nu \delta(x).
\end{aligned} \tag{37}$$

The last term is the contribution from the ν zero modes. For integer values of ν the integration contour C_c can be replaced by the segment $[-\pi, \pi]$. The restriction of the integration over θ to this segment for non-integer values of ν would have resulted in the wrong answer. This can be seen from the following representation of modified Bessel functions,

$$I_{\nu}(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{z \cos \theta} e^{i\nu\theta} d\theta - \frac{\sin \nu\pi}{\pi} \int_0^{\infty} e^{-z \cosh s - \nu s} ds. \tag{38}$$

The microscopic spectral density and the expression for $\Sigma(z)$ were first obtained from chiral Random Matrix Theory by means of the orthogonal polynomial method [31, 32] and can also be obtained by means of the supersymmetric method [33]. In that case the derivation, starting from the joint probability distribution of the eigenvalues, is also correct for non-integer values of ν and is given by the expressions (36) and (37).

The asymptotic expansion of the imaginary part of $\Sigma(z)$ is given by

$$\begin{aligned} i\text{Im}\Sigma(z) &= -\frac{i(-1)^\nu e^{-2z}}{2z} + \frac{-i(4\nu^2 - 1)(-1)^\nu e^{-2z}}{8z^2} - \frac{i(-1)^\nu e^{-2z}(4\nu^2 - 1)(4\nu^2 - 9)}{64z^3} \\ &+ \frac{i(4\nu^2 - 1)(4\nu^2 - 9)(19 - 4\nu^2)(-1)^\nu e^{-2z}}{3 \times 256z^4}, \end{aligned} \quad (39)$$

and for the asymptotic expansion of the spectral density we find

$$\begin{aligned} \rho_{s\nu}(x) &= \frac{1}{\pi} \left[1 - \frac{\cos(2x - \pi\nu)}{2x} + \frac{1 - 4\nu^2}{8x^2} (1 - \sin(2x - \pi\nu)) + (4\nu^2 - 1)(4\nu^2 - 9) \frac{\cos(2x - \pi\nu)}{64x^3} \right. \\ &\left. + (4\nu^2 - 1)(4\nu^2 - 9) \frac{(-6 + (19 - 4\nu^2) \sin(2x - \pi\nu))}{x^4 2^7 3!} + \dots \right]. \end{aligned} \quad (40)$$

The asymptotic expansion of the partition function, the resolvent, and the spectral density terminates for half-integer values of ν . For example, for $\nu = 1/2$ only one oscillating term in (40) is nonvanishing suggesting that it can be obtained from a leading order saddle point approximation.

Since for half-integer ν the integral is localized on the critical points, we expect that the asymptotic expansion generated by the replica trick reproduces the exact answer. For other values of ν we expect that the replica trick reproduces the asymptotic series to all orders in $1/x$.

4 Spectral Density via Fermionic Replicas

In this section, we analyze the low-energy partition in the quenched case ($N_f = 0$). For n replica flavors in the sector of topological charge ν it is given by (see (12)) [20]

$$Z_\nu^{(n)}(x) = \int_{U \in U(n)} dU \det^\nu U e^{\frac{x}{2} \text{Tr}(U + U^{-1})}. \quad (41)$$

In order to take the replica limit, $n \rightarrow 0$, the n -dependence of (41) has to be known explicitly. The closed form of $Z_\nu^{(n)}(x)$, in terms of a determinant of modified Bessel functions,

$$Z_\nu^{(n)}(x) = \det(I_{\nu+j-i}(x)), \quad i, j = 1, \dots, n, \quad (42)$$

does not provide us with an explicit n -dependence. The explicit n -dependence can only be obtained for the large-mass and the small-mass expansion of $Z_\nu^{(n)}(x)$. It was obtained

in [24, 25] for an expansion about the replica symmetric saddle point using the method of Virasoro constraints. Our new result is the asymptotic expansion of the microscopic spectral density using replica symmetry breaking à la Kamenev and Mézard [13] which will be discussed in the second half of the next subsection. The small mass expansion of (42) was also considered in [24, 25], but failed at the order for which logarithmic terms enter in the expansion. In the second subsection we derive the lowest order logarithmic term for the case of zero topological charge.

4.1 Large Mass Expansion

The asymptotic expansion of the microscopic spectral density is obtained from the large mass expansion of the finite volume partition function. To this end we expand the partition function (41) in powers of $1/x$ by means of a saddle point approximation. By diagonalizing U it can be easily seen that the saddle points are given by unitary matrices with eigenvalues ± 1 , i.e. by unitary matrices satisfying $U^2 = 1$. The solutions of this equation are highly degenerate. They can be organized in $n+1$ classes, $U = I_p$, where I_p is a diagonal matrix with p elements -1 and $n-p$ elements $+1$ (with $0 \leq p \leq n$). The integrand does not depend on the submanifold $U(n)/U(n-p) \times U(p)$ of $U(n)$ and the integration over this coset has to be performed exactly resulting in its volume $V_{n,p}$. In terms of the parameterization

$$U = I_p U_0 V U_0^{-1}, \quad (43)$$

with $U_0 \in U(n)/U(n-p) \times U(p)$ it is clear that the integration can be restricted to $U(n-p) \times U(p)$. Summing over all saddle points the partition function is given by

$$Z_\nu^{(n)}(x) = \sum_{p=0}^n V_{n,p} \int_{V \in U(n-p) \times U(p)} dV J(V) \det^\nu V e^{\frac{x}{2} \text{Tr}[I_p(V+V^{-1})]}. \quad (44)$$

where $V_{n,0} = 1$ and for $p \neq 0$

$$V_{n,p} = (2\pi)^{p(n-p)} \binom{n}{p} \frac{\prod_{j=1}^p j! \prod_{j=1}^{n-p} j!}{\prod_{j=1}^n j!} \equiv (2\pi)^{p(n-p)} F_n^p. \quad (45)$$

The integration over V should be thought of a saddle-point integral of a formal expansion of V about the identity to all orders. Below we will make this explicit for the different types of saddlepoints. The total number of saddle-points in the class p is $\frac{n!}{(n-p)!p!}$. This factor is included as combinatorial factor in $V_{n,p}$. The volume of the coset $U(n)/U(n-p) \times U(p)$ is given by the ratio $\frac{V(U(n))}{V(U(p))V(U(n-p))}$. With the volume of $U(k)$ given by $(2\pi)^{\frac{k(k+1)}{2}} \prod_{j=1}^k j!$ we obtain the volume factor $V_{n,p}$.

From the exact expression of the partition function (41) in terms of modified Bessel function given in (42) it is clear that the asymptotic series of $Z_n^\nu(x)$ terminates for half

integer ν . Let us investigate the asymptotic expansion about the saddle point I_p in more detail. Because the total number of degrees of freedom in $U(n-p) \times U(p)$ is equal to $(n-p)^2 + p^2$, the expansion of $Z_\nu^{(n)}(x)$ is of the form

$$Z_\nu^{(n)}(x) \sim e^{(n-2p)x} \left(\frac{1}{x}\right)^{(n-p)^2/2 + p^2/2} \left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right). \quad (46)$$

This result is valid for arbitrary ν . The asymptotic series of the Bessel function $I_{k+\frac{1}{2}}(x)$ terminates at $1/x^{k+1/2}$. The result for the maximum power in $1/x$ occuring the expansion of the determinant (42) is particularly simple for $\nu = \frac{1}{2}$ and is given by

$$\left(\frac{1}{x}\right)^{n^2/2}. \quad (47)$$

Let us consider this case in more detail. Since, as we will see below, only the saddle points for $p = 0$ or $p = 1$ contribute in the replica limit, we only discuss these values of p . We observe that the maximum power and the minimum power in the asymptotic series are equal for $p = 0$ saddle point. The asymptotic series thus has only one term and we find that

$$\left(Z_{\nu=\frac{1}{2}}^{(n)}(x)\right)_{p=0} \sim e^{nx} \left(\frac{1}{x}\right)^{n^2/2}. \quad (48)$$

For $p = 1$ and larger half-integer values of ν , more terms contribute to the asymptotic series, but it still terminates. For example, for $p = 1$ and $\nu = \frac{1}{2}$ one finds from (46) and (47) that the difference between the maximum and minimum power in the expansion in $1/\sqrt{x}$ is $n - 1$. Therefore, the asymptotic series for $Z_n^\nu(x)$ cannot contain more than n terms. From numerical examples for small values of n , one indeed finds that in this case the coefficients of all n possible terms are nonvanishing. For the same reason as in the case of the GUE two-point function [14], we expect that the replica trick will give the exact result both for $\nu = \frac{1}{2}$ as well as larger half-integer values of ν .

Clearly, the expression (44) makes only sense for positive integer values of n . In order to analytically continue it we follow the work of Kamenev and Mézard [13]. They analytically continued the factorials in F_n^p such that F_n^p vanishes for $p \geq n + 1$. Then the sum in (44) can be extended up to infinity and the replica limit $n \rightarrow 0$ can be taken term by term in the sum over p . One finds

$$\lim_{n \rightarrow 0} F_n^p \sim n^p, \quad (49)$$

so that only the terms $p = 0$ and $p = 1$ of the, to infinity continued, sum in (44) survive. In these two cases we obtain $F_{n \rightarrow 0}^0 = 1$, $F_{n \rightarrow 0}^1 = n$. Notice, that the continuation of the sum over p to infinity explicitly breaks the replica symmetry $p \rightarrow n - p$. Of course, the group integral in (30) must be also continued to non-integer values of n .

The previous discussion suggests the definition

$$Z_\nu^{(n)}(x) \equiv \left(Z_\nu^{(n)}(x) \right)_{p=0} + n \left(Z_\nu^{(n)}(x) \right)_{p=1}. \quad (50)$$

We will first consider the contribution for $p = 0$ which originates from expanding U around the identity matrix I_0 . Although not necessary, it turned out to be convenient to parameterize $V_{p=0} \in U(n)$ according to

$$V_{p=0} = \frac{1 + iH/2}{1 - iH/2}, \quad (51)$$

where H is an Hermitian $n \times n$ matrix. From a diagonal representation of $V_{p=0}$ one can easily show that [34]

$$dV_{p=0} = \frac{1}{\det^n(1 + H^2/4)} dH. \quad (52)$$

In the replica limit, the Jacobian can be ignored and one simply has $dV_{p=0} = dH$. The $p = 0$ contribution to the partition function is thus given by

$$\left(Z_\nu^{(n)}(x) \right)_{p=0} = \int dH \det \left(\frac{1 + iH/2}{1 - iH/2} \right)^\nu e^{nx - \frac{x}{2} \text{Tr} H^2 + x \text{Tr} \left(\frac{H^4/8}{1 + H^2/4} \right)}. \quad (53)$$

In order to keep track of the powers of x it is convenient to rescale $H \rightarrow H/\sqrt{x}$ which leads to another Jacobian $x^{-n^2/2}$ that also vanishes in the replica limit. In terms of $x = m\Sigma_0 V$, the mass dependence of the condensate is given by

$$\frac{\Sigma_{p=0}(x)}{\Sigma_0} = \lim_{n \rightarrow 0} \frac{1}{n} \frac{1}{Z_\nu^{(n)}(x)} \frac{\partial}{\partial x} \left(Z_\nu^{(n)}(x) \right)_{p=0}. \quad (54)$$

so that to order $1/x^4$ we need to collect terms to order $1/x^3$ in the expansion of the partition function. Using the expressions

$$\begin{aligned} e^{\frac{x}{2} \text{Tr}(V_{p=0} + V_{p=0}^{-1})} &= e^{nx - \frac{1}{2} \text{Tr} H^2} e^{\text{Tr} \left[\frac{1}{8x} H^4 - \frac{1}{32x^2} H^6 + \frac{1}{128x^3} H^8 - \dots \right]} \\ (\det V_{p=0})^\nu &= e^{\nu n} e^{\frac{i\nu \text{Tr}}{\sqrt{x}} \left(H - \frac{2H^3}{3 \times 2^3 x^2} + \dots \right)}. \end{aligned} \quad (55)$$

we find the result (only terms up to order $1/x^2$ are displayed)

$$\begin{aligned} \left(Z_\nu^{(n)} \right)_{p=0} &= \frac{e^{nx + \nu n}}{x^{\frac{n^2}{2}}} \int dH e^{-\frac{1}{2} \text{Tr} H^2} \\ &\times \left(1 - \frac{\nu^2}{2x} (\text{Tr} H)^2 + \frac{\nu^2}{12x^2} (\text{Tr} H)(\text{Tr} H)^2 + \frac{\nu^4}{24x^2} (\text{Tr} H)^4 \right) \\ &\times \left(1 + \frac{1}{8x} \text{Tr} H^4 - \frac{1}{32x^2} \text{Tr} H^6 \right) \\ &= \left(\frac{2\pi}{x} \right)^{n^2/2} e^{nx} \left(1 - n \frac{\nu^2}{2x} + (2n^3 + n) \frac{1}{8x} \right) + \mathcal{O}\left(\frac{n^2}{x^2}\right). \end{aligned} \quad (56)$$

The Gaussian integrals have been calculated using the trace correlators given in the Appendix. To this order, the Jacobian in (52) gives rise to an extra factor $(1 - n^3 x/4)$. With inclusion of this terms the $1/x$ corrections vanish for $\nu = \frac{1}{2}$. All terms of order $1/x^2$ are at least of order n^2 . The resolvent obtained from (54)

$$\frac{\Sigma_{p=0}(x)}{\Sigma_0} = 1 + \frac{4\nu^2 - 1}{8x^2} - \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{128x^4} + \dots \quad (57)$$

agrees with the large mass expansion of the quenched condensate obtained via other methods [24]. We observe a cancellation of the odd powers in $1/x$. This is in agreement with the analytical expression for $\Sigma(x)$ given in (36) which can be rewritten as

$$\frac{\Sigma(x)}{\Sigma_0} = \frac{x}{2} \left((2 - \frac{4\nu^2}{x^2}) I_\nu(x) K_\nu(x) + I_{\nu+1}(x) K_{\nu+1}(x) + I_{\nu-1}(x) K_{\nu-1}(x) \right). \quad (58)$$

From the asymptotic behavior of the Bessel functions, one easily derives that the asymptotic series of this expression is an expansion in powers of $1/x^2$. Also notice the cancellation of the ν/x term in the large- x asymptotic expansion.

Next we consider the more subtle contribution of the saddle point given by the diagonal matrix I_1 with one element equal to -1 and $n-1$ elements equal to 1 . A parameterization of $U(n-1) \times U(1)$ that is convenient for the expansion about the saddle point is given by

$$V_{p=1} = \begin{pmatrix} \frac{1+iH/2}{1-iH/2} & \\ & \frac{1+ih/2}{1-ih/2} \end{pmatrix}, \quad (59)$$

where, now, H is a hermitian $(n-1) \times (n-1)$ matrix and h is a real variable. Because $U_0 \in U(n)/U(n-1) \times U(1)$ we have that

$$U = I_1 U_0 V_{p=1} U_0^{-1} = U_0 I_1 V_{p=1} U_0^{-1}. \quad (60)$$

The measure, in terms of the coordinates H and h , can be obtained by diagonalizing U and H with unitary transformations U_1 and W , respectively. If the eigenvalues are denoted by $e^{i\theta}$ and h_k , in this order, we find the measure (notice the plus sign in the last factor)

$$\begin{aligned} dU &= \prod_{k < l} |e^{i\theta_k} - e^{i\theta_l}|^2 \prod_k d\theta_k dU_1 \\ &= \frac{\prod_{k < l} |h_k - h_l|^2}{(1 + h^2/4) \prod_k (1 + h_k^2/4)^{n-1}} \prod_k \left| \frac{1 + ih_k/2}{1 - ih_k/2} + \frac{1 + ih/2}{1 - ih/2} \right|^2 dh \prod_k dh_k dU_0 dV. \end{aligned} \quad (61)$$

The last factor can be written as

$$\prod_k \left| \frac{1 + ih_k/2}{1 - ih_k/2} + \frac{1 + ih/2}{1 - ih/2} \right|^2 = 4^{n-1} \frac{\det^2(1 + hH)}{(1 + h^2/4)^{n-1} \det(1 + H^2/4)}. \quad (62)$$

Using that $\prod_{k < l} |h_k - h_l|^2 \prod_k dh_k dV = dH$ we thus find the measure

$$dU = 4^{n-1} dU_0 \frac{dh dH \det^2(1 + hH)}{(1 + h^2/4)^n \det^n(1 + H^2/4)} \equiv J(H, h) dh dH dU_0. \quad (63)$$

The integrand does not depend on U_0 and the integration over these variables just gives the volume of the coset which, together with the combinatorial factor, combines into the factor $V_{n,1}$ discussed in the first part of this subsection. The $p = 1$ contribution to the partition function is thus given by (with one overall minus sign from orienting the h -integration from $-\infty$ to ∞)

$$\begin{aligned} \left(Z_\nu^{(n)}(x) \right)_{p=1} &= -(-1)^\nu (8\pi)^{n-1} \int dH dh \frac{\det^2(1 + hH)}{(1 + h^2/4)^n \det^n(1 + H^2/4)} \\ &\times \left(\frac{1 + ih/2}{1 - ih/2} \right)^\nu \det \left(\frac{1 + iH/2}{1 - iH/2} \right)^\nu e^{x \text{Tr} \left(\frac{1 - H^2/4}{1 + H^2/4} \right) - x \left(\frac{1 - h^2/4}{1 + h^2/4} \right)}. \end{aligned} \quad (64)$$

Since the factors in the denominator of the Jacobian can be ignored in the replica limit only the following terms in the expansion of the Jacobian contribute to order $1/x^4$,

$$J = 4^{n-1} \left[1 + \frac{h \text{Tr} H}{2} + \frac{h^2}{16} [2(\text{Tr} H)^2 - \text{Tr} H^2] + \dots \right]. \quad (65)$$

It is instructive to perform the calculation to leading order in $1/x$. In this case the following terms should be collected

$$\begin{aligned} \left(Z_\nu^{(n)}(x) \right)_{p=1} &= -(-1)^\nu (8\pi)^{n-1} e^{(n-2)x + \nu n} \int dH dh e^{-\frac{x}{2} \text{Tr} H^2 + \frac{x}{2} h^2} \\ &\times \left[1 + \frac{x}{8} \text{Tr} H^4 - \frac{x}{8} h^4 - \frac{\nu^2}{2} ((\text{Tr} H)^2 + h^2 + 2ih \text{Tr} H) + \frac{h \text{Tr} H}{2} \right]. \end{aligned} \quad (66)$$

The last two terms vanish upon integration. Thus, the Jacobian contributes only at the next to the leading order. The saddle-point integrations can be performed conveniently by rescaling h and H according to $h \rightarrow h/\sqrt{-x}$; $H \rightarrow H/\sqrt{x}$. This results in

$$\begin{aligned} \left(Z_\nu^{(n)}(x) \right)_{p=1} &= i(-1)^\nu (8\pi)^{n-1} \left(\frac{2\pi}{x} \right)^{((n-1)^2+1)/2} e^{(n-2)x + \nu n} \\ &\times \left[1 + (2(n-1)^3 + (n-1)) \frac{1}{8x} - \frac{3}{8x} - n(n-2) \frac{\nu^2}{2x} \right]. \end{aligned} \quad (67)$$

Now we are in a position to calculate the contribution of the $p = 1$ saddle point to the chiral condensate,

$$\frac{\Sigma_{p=1}(x)}{\Sigma_0} = \lim_{n \rightarrow 0} \frac{1}{n} \frac{1}{Z_\nu^n(x)} \frac{\partial}{\partial x} [n \left(Z_\nu^{(n)}(x) \right)_{p=1}]. \quad (68)$$

Using trace correlators given in the Appendix, the expansion in $1/x$ can be easily extended to order $1/x^4$. For the mass dependence of the chiral condensate we obtain

$$\begin{aligned}
\frac{\Sigma(x)}{\Sigma_0} &= \lim_{n \rightarrow 0} \frac{1}{n} \frac{1}{Z_\nu^n(x)} \frac{\partial}{\partial x} Z_\nu^n(x) \\
&= 1 - \frac{i(-1)^\nu e^{-2x}}{2x} + \frac{(4\nu^2 - 1)(1 - i(-1)^\nu e^{-2x})}{8x^2} - \frac{i(-1)^\nu e^{-2x}(4\nu^2 - 1)(4\nu^2 - 9)}{64x^3} \\
&\quad + \frac{(4\nu^2 - 1)(4\nu^2 - 9)[i(-1)^\nu e^{-2x}(19 - 4\nu^2) - 6]}{3 \times 256x^4}.
\end{aligned} \tag{69}$$

All terms $\sim ie^{-2x}$ originate from $p = 1$ saddle point. Before calculating the spectral density by taking the discontinuity of $\Sigma(x)$ we wish to point out that the expression (69) has been obtained under the assumption that $\text{Re}(x) > 0$, so that we cannot calculate the discontinuity from the difference of $\Sigma(i\lambda + \epsilon)$ and $\Sigma(i\lambda - \epsilon)$. The reason is that for $x = i\lambda + \epsilon$ the dominant saddle-point is given by I_0 and for $x = i\lambda - \epsilon$ it is given by $-I_0$. The infinitesimal increment thus breaks the replica symmetry between I_0 and $-I_0$. The sum over p in (44) has been extended to ∞ consistent with the breaking of the replica symmetry $p \rightarrow n - p$ by the saddle point I_0 . Thus, the infinitesimal increment is necessary to resolve the ambiguity as was also the case in the original calculation of [13]. The expression for $\Sigma(i\lambda - \epsilon)$ can be obtained from a replica-symmetry breaking solution for which $-I_0$ dominates. The final result for the asymptotic expansion of $\rho_s(\lambda)$ coincides with (40) obtained from the expansion of the analytical result [31] $(\lambda/2)(J_\nu^2(\lambda) - J_{\nu+1}(\lambda)J_{\nu-1}(\lambda))$ up to $1/\lambda^4$ for $\lambda > 0$.

We observe that it requires a great deal of effort to derive the asymptotic expansion of the oscillating contribution to the spectral density by means of the replica trick. This is especially true due to the lack of the Virasoro constraints for $\nu \neq 0$. Those constraints are a key tool [24] in simplifying the calculations for the mass expansions of the condensate in the sector of vanishing topological charge ($\nu = 0$). The simplifying feature in the sector with vanishing topological charge that the partition function can be shown to belong to the universality class (see [22]) of the generalized Kontsevich model with potential $\mathcal{V}(x) = 1/x^2$ and satisfies the same Virasoro constraints. For $\nu \neq 0$ the one-link integral depends also on $\det J$ and $\det J^\dagger$ which is also the case when we have integrals over $SU(N_f)$ instead of $U(N_f)$. Therefore, a possible generalization of the Virasoro constraints to include the case $\nu \neq 0$ would be certainly welcome not only to reduce our efforts in perturbative calculations but also as an identification of the universality class of the QCD finite volume partition function at nonzero topological charge.

4.2 Small Mass Expansion

The situation for the small mass is much more complicated. Before discussing the complications we recall that the partition function $Z_\nu^n(x)$ has been extensively studied in the

context of lattice QCD where it is known as the one-link integral (see for example [37] for an updated review of the subject). In that context one considers unitary matrix integrals of the form

$$Z(J, J^\dagger) = \int_{U \in U(n)} dU e^{\text{Tr}(JU^\dagger + J^\dagger U)}, \quad (70)$$

where J is a general $n \times n$ matrix. The partition function with such potential is a function of the eigenvalues λ_k of the matrix JJ^\dagger . The one-link model exhibits two phases according to [23]

$$\begin{aligned} \sum_{k=1}^n \frac{1}{2\sqrt{\lambda_k}} &\leq 1 \quad (\text{weak coupling}), \\ \sum_{k=1}^n \frac{1}{2\sqrt{\lambda_k}} &\geq 1 \quad (\text{strong coupling}). \end{aligned} \quad (71)$$

In our case, with partition function given by (42), the eigenvalues of $J^\dagger J$ are given by $x^2/4$. From the above we see that, for $x \rightarrow \infty$, we expect to take the replica limit, $n \rightarrow 0$, and remain in the weak coupling regime while, for $x \rightarrow 0$, it is not clear whether the replica limit can be taken without crossing a phase boundary. These problems are reflected in logarithmic singularities of the small x expansion of the valence quark mass dependence of the partition function and its derivatives. In [24] the replica limit of the expansion coefficients could be derived up to the order for which terms of the form $x^p \log x$ are absent. These singular terms could be related [24] to the presence of de Wit-'t Hooft poles [38].

Below we consider the replica limit of $Z_\nu^n(x)$ for $\nu = 0$. In that case logarithmic terms already enter to lowest order in the expansion. We thus consider the small x expansion of the partition function $Z_{\nu=0}^n(x) = \det(I_{i-j}(x))$. By expanding the Bessel functions we obtain

$$Z_0^n(x) = \sum_{k=0}^n \left(\frac{x^2}{4}\right)^k \frac{1}{k!} + \sum_{k=n+1}^{\infty} C_{k,n} x^{2k}. \quad (72)$$

The first sum can be recognized as an incomplete exponential, but we were not able to determine the coefficients $C_{k,n}$ in general. In order to expose the n -dependence of the first term, we rewrite the incomplete exponential as an incomplete gamma function

$$\sum_{k=0}^n \left(\frac{x^2}{4}\right)^k \frac{1}{k!} = \frac{e^{\frac{x^2}{4}} \Gamma(n+1, x^2/4)}{\Gamma(n+1)}. \quad (73)$$

From the small mass expansion of the incomplete gamma function

$$\Gamma(n+1, x^2/4) = \int_{x^2/4}^{\infty} dt t^n e^{-t} = \Gamma(n+1) + \sum_{k=1}^{\infty} \left(\frac{x^2}{4}\right)^{k+2n} \frac{(-1)^k}{(n+k)(k-1)!}, \quad (74)$$

and neglecting terms which will vanish in the replica limit, i.e.,

$$\begin{aligned}\Gamma(n+1) &= 1 - \gamma n + \mathcal{O}(n^2) \\ \left(\frac{\mu^2}{4}\right)^n \frac{1}{n+k} &= \frac{1}{k} + n \left(\frac{2}{k} \log \frac{\mu}{2} - \frac{1}{k^2}\right) + \mathcal{O}(n^2),\end{aligned}\tag{75}$$

we find

$$\begin{aligned}Z_0^n(x) &= 1 + n \left[(1 - e^{x^2/4}) (2 \log \frac{x}{2} + \gamma) + e^{x^2/4} \sum_{k=1}^{\infty} \left(-\frac{x^2}{4}\right)^k \frac{1}{k!k} \right] + \mathcal{O}(n^2) \\ &+ \sum_{k=n+1}^{\infty} C_{k,n} x^{2k}.\end{aligned}\tag{76}$$

Because the coefficients $C_{k,n}$ are unknown only the lowest order terms of the small mass expansion can be calculated,

$$Z_0^n(x) = 1 - n \frac{x^2}{2} \log x + \mathcal{O}(x^2) + \mathcal{O}(n^2).\tag{77}$$

For the replica limit of the condensate given by,

$$\Sigma(x) = \lim_{n \rightarrow 0} \frac{1}{n Z_0^n(x)} \frac{\partial Z_0^n(x)}{\partial x},\tag{78}$$

we thus obtain

$$\Sigma(x) = -x \log x + \mathcal{O}(x^1).\tag{79}$$

This is indeed the correct leading order term of the small mass expansion of $x(I_0(x)K_0(x) + I_1(x)K_1(x))$. The linear behavior of the microscopic spectral density at the origin is reproduced by taking the discontinuity across the imaginary axis,

$$\rho_s(\lambda) = \frac{\lambda}{2} + \mathcal{O}(\lambda^2).\tag{80}$$

We did not succeed to generalize this calculation to arbitrary ν , but we expect that the logarithmic terms can be obtained in a similar fashion. In particular, the first $n + \nu$ terms of the expansion seem to follow a simpler pattern than the coefficients of the higher powers.

5 Bosonic Replicas

In view of the seemingly different role [7, 14] played by bosonic and fermionic replicas and the fact that the supersymmetric method uses both compact and noncompact variables

it is natural to try to reproduce the results of last section by introducing additional n replicas of bosonic quarks of mass m instead of fermionic ones. In this case, the condensate for $N_f = 0$ is given by

$$\Sigma(m) = \lim_{n \rightarrow 0} \frac{1}{(-n)} \frac{1}{V_4} \frac{\partial}{\partial m} \ln Z_\nu^{(-n)}(x), \quad (81)$$

where $Z_\nu^{(-n)}(x)$ is given in (4). We will show that, for large masses, bosonic replicas can be used to reproduce the asymptotic expansion of the chiral condensate but they fail to reproduce the microscopic spectral density for a subtle reason which we will explain below.

It is convenient to express the bosonic partition function $Z_\nu^{(-n)}(x)$ in terms of the eigenvalues of the $Gl(n)/U(n)$ matrices. Up to an overall constant we have,

$$Z_\nu^{(-n)}(x) = \int_{-\infty}^{\infty} \prod_{k=1}^n ds_k \prod_{k < j} (e^{s_k} - e^{s_j})(e^{-s_k} - e^{-s_j}) e^{-x \sum_{k=1}^n \cosh s_k + \nu \sum_{k=1}^n s_k}. \quad (82)$$

Notice in particular that the measure (including the Vandermonde determinant) is invariant under the symmetry $s_k \rightarrow s_k + t$ which is a remnant of the $Gl(n)/U(n)$ invariance. The fermionic partition function $Z_\nu^{(n)}(x)$ is given by a circular ensemble (for integer ν) [20],

$$Z_\nu^{(n)}(x) = \int_{-\pi}^{\pi} \prod_{k=1}^n d\theta_k \prod_{k < j} (e^{i\theta_k} - e^{i\theta_j})(e^{-i\theta_k} - e^{-i\theta_j}) e^{x \sum_{k=1}^n \cos \theta_k + i\nu \sum_{k=1}^n \theta_k}. \quad (83)$$

In the noncompact case, the solutions of the saddle point equation $\sinh s_k = 0$ are given by $s_k = 0, \pm i\pi, \pm i2\pi \dots$. Thus, in principle we might have a variety of saddle points which should be all taken into account in large x expansion. However, we will argue that only $s_k = 0$ solution contributes to the large x behavior of $Z_\nu^{(-n)}(x)$. Our discussion is based on the $n = 1$ integral where a steepest descent analysis can be easily carried out. In this case, the bosonic partition function is given by

$$Z_\nu^{(-1)}(x) = \int_{-\infty}^{\infty} ds e^{-x \cosh(s) + \nu s} = 2K_\nu(x) \quad (84)$$

From the asymptotic expansion of $K_\nu(x)$ it is clear that only the saddle-point at $s = 0$ contributes to the integral. The spectral density can be calculated from the resolvent at $x = \pm i\lambda + \epsilon$ so that the integral above can be identified with the modified Bessel function $K_\nu(x)$.

On the other hand, for one fermionic replica the partition function is given by (assuming integer ν)

$$Z_\nu^{(1)}(x) = 2\pi I_\nu(x). \quad (85)$$

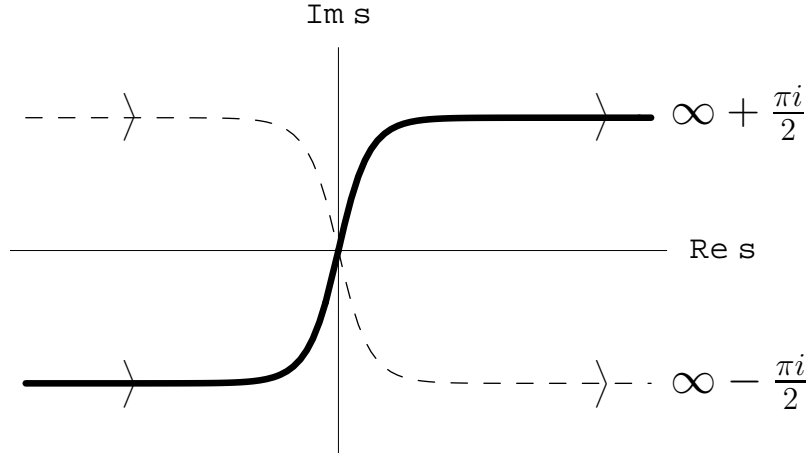


Figure 2: Steepest descent curve passing through the saddle point $s = 0$ for $\lambda < 0$ (solid) and for $\lambda > 0$ (dashed), where we assumed $x = i\lambda + \epsilon$.

In this case two saddle-points contribute to the asymptotic expansion, one $\sim e^x$, and the other one $\sim e^{-x}$, which are both essential to recover the oscillatory contributions to the microscopic spectral density.

Let us analyze in detail the saddle-point calculation of the bosonic partition function for one replica. The stationary phase condition that the imaginary part of the action, $-i\lambda \cosh s$, is constant results in the following curve in the complex s -plane through the saddle point at $s = 0$. Clearly, the integration contour (the real axis) can be deformed into the steepest descent curve of fig. 2 (depending on the sign of λ) and no other saddle points need to be considered. A similar analysis for the compact case shows that both the saddle points at $\theta = 0$ and at $\theta = \pi$ have to be taken into account.

A similar situation arises in the $1/N$ correction to the semicircle law of the Gaussian Unitary Ensemble (GUE) of hermitian matrices. Inside the semicircle the saddle point solutions of the $n = 1$ fermionic replica are horizontally aligned in the complex plane, while (see also a comment in [14]) the solutions for the one bosonic replica are vertically aligned and only the $p = 0$ saddle point contributes. If we insist on taking into account all vertically aligned saddle points we get an incorrect result. However, we have checked that outside the semicircle the situation is reversed (see figures in [35] for the fermionic case). In this case the bosonic replicas correctly reproduce the semicircle law and its leading exponentially decreasing correction outside the semicircle (see also [36]) while the fermionic replica calculation gives an incorrect result. In the case of the microscopic spectral density analyzed in this work, we are always inside the semicircle even in the limit $x \rightarrow \infty$ which explains, assuming the same pattern of as for the GUE, the failure of

the bosonic replica calculation in reproducing the oscillating part of the spectral density.

If we are just interested in the mass dependence of the chiral condensate it is sufficient to only take into account the $p = 0$ saddle point and, one can convince oneself that bosonic and fermionic replicas produce the same large x result as follows. The integrand of the non-compact partition function is obtained by transforming the integration variables according to $\theta_k = i s_k$ and replacing $x \rightarrow -x$. For large x , the saddle-point of both partition functions is at $\theta_k = 0$ and $s_k = 0$, respectively, and one finds that $Z_\nu^{(n)}(-x)$ and $Z_\nu^{(-n)}(x)$ have the same asymptotic expansion for the chiral condensate (see (57)). Clearly this proof is formal because it assumes that the replica limit can be interchanged with the operation $x \rightarrow -x$. However, we have explicitly checked to the order $1/x^3$ that the noncompact partition function $Z_\nu^{(-n)}(x)$ leads to the same asymptotic expansion.

6 Conclusions

We have investigated the replica trick for the microscopic spectral density of the QCD Dirac operator in the quenched limit. The advantage of working with fermionic replicas is that this theory corresponds to QCD with n flavors of equal mass m . Because the low energy properties of this theory have are well understood, the starting point of this approach has a firm basis. The valence quark mass dependence of the chiral condensate and the spectral density of the QCD Dirac operator, however, are only obtained in the limit $n \rightarrow 0$. The existence of this limit has been debated for many years and the investigation of its nature has been the main topic of this article.

The alternative approach to obtain the QCD Dirac spectrum is the supersymmetric method. Although in principle rigorous, one might raise the question whether this theory with bosonic ghost quarks might have unusual properties as for example the spontaneous breaking of supersymmetry. Our results show that this is not the case. The low-energy limit of this partition function is completely dictated by chiral supersymmetry. The power of the supersymmetric method is that one obtains rigorous nonperturbative results such as, for example, the spectral density in the microscopic region.

The replica trick, on the other hand, requires an explicit n dependence, and, up to now, only perturbative results have been obtained. Exact results have only been derived in cases where the perturbative series consists of only a finite number of terms. Our results for the microscopic spectral density confirm that the asymptotic series of its non-oscillatory part can be obtained from an expansion about the replica symmetric saddle-point. Both fermionic and bosonic replicas give the same result and are in complete agreement with the asymptotic expansion of the exact result. The same is true for the asymptotic expansion of the resolvent away from the imaginary axis (where the eigenvalues are located). Things are different for the the oscillatory part of the microscopic spectral density. In this case

the correct asymptotic expansion is obtained only if a saddle point that breaks the replica symmetry is taken into account. This additional saddle point only exists for fermionic replicas. For bosonic replicas, only the replica symmetric saddle point contributes in the saddle point calculation, and therefore this approach does not reproduce the asymptotic expansion of the oscillatory part of the spectral density. A similar observation has been made for the application of the replica trick to the Wigner-Dyson ensembles.

In the asymptotic expansion of the supersymmetric partition function also two saddle-points have to be taken into account. In the boson-boson component of the Goldstone manifold only one saddle point contributes, but as is the case for one fermionic replica, we have to take into account two saddle-points in the fermion-fermion component of the Goldstone manifold.

Is it possible to go beyond perturbation theory using the replica trick? One indication in favor of an affirmative answer to this question is that we have reproduced the leading order logarithmic singularity of the small mass expansion of the resolvent. Except for these logarithmic terms, the small mass expansion of the resolvent is a convergent series which may be summed to obtain the exact result. The large mass expansion, on the other hand, is an asymptotic series which cannot be summed and cannot provide us with the exact result. Of course, for half integer ν when the asymptotic series terminates, an exact result is obtained from the replica trick.

The exact answer for the resolvent in the microscopic region shows a compact-noncompact dichotomy. This dichotomy is natural in the supersymmetric approach where the compact part of the Goldstone manifold is associated with the fermion-fermion sector and the non-compact part is associated with the boson-boson sector. In the fermionic replica trick this dichotomy is not at all clear but might be hidden in the $n \rightarrow 0$ limit which is given by an integration over a 1×1 matrix and an integral over an $(n-1) \times (n-1)$ matrix. This might be another hint that it is possible to go beyond perturbation theory within the replica framework.

Finally, we hope to have convinced the reader that the supersymmetric method is the only σ -model approach that can provide us with rigorous exact results. Even the calculation of a small number of terms in the asymptotic expansion within the replica approach requires a tremendous effort in the case of broken replica symmetry.

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Appendix

We are interested in expectation values of traces of powers of $n \times n$ Hermitian matrices with matrix elements distributed according to the Gaussian Unitary Ensemble (GUE). Such averages are given by

$$\begin{aligned}\Omega_q(p_1, p_2, \dots, p_q) &= \langle \text{Tr} H^{p_1} \text{Tr} H^{p_2} \dots \text{Tr} H^{p_q} \rangle = \frac{\int dH e^{-\frac{1}{2} \text{Tr} H^2} \text{Tr} H^{p_1} \text{Tr} H^{p_2} \dots \text{Tr} H^{p_q}}{\int dH e^{-\frac{1}{2} \text{Tr} H^2}} \\ &= \frac{\int_{-\infty}^{\infty} \prod_{i=1}^n \left(dx_i e^{-\frac{1}{2} x_i^2} \right) \prod_{i < j} (x_i - x_j)^2 \sum_{k_1=1}^n x_{k_1}^{p_1} \sum_{k_2=1}^n x_{k_2}^{p_2} \dots \sum_{k_q=1}^n x_{k_q}^{p_q}}{\int_{-\infty}^{\infty} \prod_{i=1}^n \left(dx_i e^{-\frac{1}{2} x_i^2} \right) \prod_{i < j} (x_i - x_j)^2} \\ &= \langle \omega_{p_1} \dots \omega_{p_q} \rangle,\end{aligned}\tag{86}$$

where $\omega_l \equiv \text{Tr} H^l$. All correlators $\Omega_q(p_1, \dots, p_q)$ can be calculated recursively starting from $\Omega_1(0) = n$. The recursion relations are derived from integrals of total derivatives (Schwinger-Dyson equations),

$$\int_{-\infty}^{\infty} \prod_{i=1}^n dx_i \sum_{k=1}^n \partial_k \left(x_k^{a+1} \sum_{j_1=1}^n x_{j_1}^{r_1} \dots \sum_{j_b=1}^n x_{j_b}^{r_b} \Delta_n^2 e^{-\frac{1}{2} \sum_{i=1}^n x_i^2} \right) = 0.\tag{87}$$

It is useful to keep in mind the identity

$$\sum_{k=1}^n \partial_k \left(x_k^{a+1} \Delta_n^2 \right) = \Delta_n^2 \sum_{l=1}^a \omega_l \omega_{a-l}.\tag{88}$$

As a sample calculation let us derive $\Omega_3(1, 1, 4)$. Choosing $b = 2, r_1 = 4, r_2 = 1$ and $a = -1$ in (87) we find

$$\Omega_3(1, 1, 4) = 4\Omega_2(1, 3) + n\Omega_1(4).\tag{89}$$

Choosing now $b = 1, a = -1$ and $r_1 = 3$ or $r_1 = 1$, respectively, we get

$$\begin{aligned}\Omega_2(1, 3) &= 3\Omega_1(2), \\ \Omega_2(1, 1) &= n.\end{aligned}\tag{90}$$

Finally, from $b = 0$ and $a = 2$ or $a = 0$, respectively, we deduce

$$\begin{aligned}\Omega_1(2) &= n^2, \\ \Omega_1(4) &= 2n\Omega_1(2) + \Omega_2(1, 1).\end{aligned}\tag{91}$$

This system of equations results in

$$\Omega_3(1, 1, 4) = 2n^4 + 13n^2.\tag{92}$$

For some special correlators we can easily derive a general formula, e.g., choosing $b = 2k$, $a = -1$ and $r_1 = r_2 = \dots = r_{2k} = 1$ we find

$$\Omega_{2k}(1, 1, \dots, 1) = n^k(2k-1)!! . \quad (93)$$

Clearly all $\Omega_q(p_1, \dots, p_q)$ with $\sum_i p_i$ being odd vanish identically. Besides (93) we have used the following moments in the calculation of this paper,

$$\begin{aligned} \Omega_1(0) &= n, \\ \Omega_1(2) &= n^2, \\ \Omega_1(4) &= 2n^3 + n, & \Omega_2(2, 2) &= n^4 + 2n^2, \\ \Omega_2(1, 3) &= 3n^2, & \Omega_3(1, 1, 2) &= n^3 + 2n, \\ \Omega_1(6) &= 5n^4 + 10n^2, & \Omega_2(2, 4) &= 2n^5 + 9n^3 + 4n, \\ \Omega_2(3, 3) &= 12n^3 + 3n, & \Omega_2(1, 5) &= 10n^3 + 5n, \\ \Omega_3(1, 1, 4) &= 2n^4 + 13n^2, & \Omega_4(1, 1, 1, 3) &= 9n^3 + 6n, \\ \Omega_1(8) &= 14n^5 + 70n^3 + 21n, & \Omega_2(2, 6) &= 5n^6 + 40n^4 + 60n^2, \\ \Omega_2(4, 4) &= 4n^6 + 40n^4 + 61n^2, & \Omega_3(1, 1, 6) &= 5n^5 + 70n^3 + 30n, \\ \Omega_3(1, 3, 4) &= 6n^5 + 75n^3 + 24n, & \Omega_5(1, 1, 1, 1, 4) &= 6n^5 + 75n^3 + 24n, \\ \Omega_3(2, 4, 4) &= 4n^8 + 72n^6 + 381n^4 + 488n^2, \\ \Omega_2(4, 6) &= 10n^7 + 169n^5 + 610n^3 + 156n, \\ \Omega_4(1, 1, 4, 4) &= 4n^7 + 88n^5 + 661n^3 + 192n, \\ \Omega_3(4, 4, 4) &= 8n^9 + 228n^7 + 2202n^5 + 6517n^3 + 1440n. \end{aligned} \quad (94)$$

The sum of the coefficients of the correlators can always be checked by means of the $n = 1$ case where all correlators reduce to one dimensional Gaussian integrals,

$$\Omega_q(p_1, \dots, p_q)|_{n=1} = \left(\sum_i p_i - 1 \right)!! . \quad (95)$$

The coefficient of the highest order power in n of the correlators $\Omega_1(2k)$ can be checked as follows. First, in the limit $n \rightarrow \infty$ one can solve the loop equation (or Virasoro constraints) which, for our Gaussian potential, gives the equation

$$\left\langle \sum_{i=1}^n \frac{1}{p - x_i} \right\rangle = \frac{p - \sqrt{p^2 - 4n}}{2}, \quad (96)$$

where p is a positive number assumed to be large (outside the semi-circle). After expanding both sides of this equation in powers of $1/p$ we find

$$\lim_{n \rightarrow \infty} \Omega_1(2k) = \frac{(2n)^{k+1}(2k-1)!!}{2(k+1)!}. \quad (97)$$

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